TRAVELING DENSITY WAVES IN A POSITIVE COLUMN

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ABSTRACT

The field of gaseous electronics has become one of burgeoning importance in recent years due to the vast strides made in the field of communications and other related fields. Theories abound, but few impregnable mathematical descriptions of the phenomenon of the positive column have been advanced. An attempt was made to extend one of the mathematical solutions to a more general case. The resulting equations were programmed into a digital computer which produced results indicating the efficacy of the program, but more data will have to be digested in order to establish the reliability of the basic equations.

The writers wish to express their sincere appreciation to Professor Norman L. Oleson for suggesting this problem and for encouraging us and guiding our research. We wish to thank Professor H. Martinez for his guidance and instruction on the computer portion of this thesis--without his help this thesis could not have been brought to a successful conclusion. We would also like to thank our two most invaluable assistants--our wives, whose unflagging cheerfulness, faith, and understanding made this work possible.

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CHAPTER I

INTRODUCTION

This thesis evolved from the analysis of a paper by Professors M. Satosi Watanabe and Norman L. Oleson of the U. S. Naval Postgraduate School(1) in which the phenomenon of traveling density waves in a positive column was investigated. It seems appropriate in the beginning to define exactly what is meant by some of the terms and expressions that will be used hereafter. This is done expressly to avoid confusion over semantics and not as a gratuitous collective insult to subsequent readers.

In the simplest instance let us assume that we have a cylindrical glass tube filled with a pure rare gas at a pressure of say, 12 mm Hg. In terms of every day experience this would be considered a relatively low pressure; however, most authorities in the field of gaseous conductors refer to this as a high pressure. Sealed in either end of our gas tube let us imagine that we have two electrodes which conform to the shape of the tube. Let each of these electrodes be connected to an external source of e.m.f. through an external resistor. Now, it is well established that if either the e.m.f. or the resistor is varied, a glow discharge can be induced to occur in the tube. Generally speaking, the glow discharge is a steady state gaseous discharge which appears at reduced pressures and is characterized by relatively low current densities and relatively high potential gradients; perhaps it is better to say in this latter case, relatively high values of $\frac{X}{p}$, electric field to pressure ratios, about which more will be said later. This gaseous discharge has a typical structure which is marked by a rather steep potential gradient at the

cathode and operates basically by electron liberation resulting from positive ion bombardment at the cathode. Quite simply, this amounts to nothing more than having the gas act as a conductor. However, the phenomena occurring within the gas while it is acting as a conductor cannot be so simply explained. In fact so many phenomena occur that it is not feasible to discuss all of them within the scope of this thesis. As indicated in the opening statement, we shall confine our attention to a single phenomenon in the positive column of the glow discharge. In order to facilitate our development we shall make some simplifying assumptions about the composition of our positive column. We shall assume that it is composed of three fluids:

a. A negative ion "gas" which we shall further assume to be composed entirely of electrons

b. A positive ion "gas" which we shall further assume to be composed either of atoms or molecules of the gas with which the tube is filled, but each of which carries a single positive charge

c. The neutral particles of the gas itself--these particles are by far in the majority.

It is customary in discussing gaseous conductors to refer to ionized regions containing high and approximately equal concentrations of positive and negative ions as the plasma(2). With the proper environment there can be caused to appear in the positive column alternate dark and light bands. These bands may move or they may be stationary. In either case they are called striations. If they appear to move, they are called moving

striations; if they do not move, they are called *standing striations*. As a parenthetical matter of interest, it seems well to note that moving striations are not detectable by the unaided eye.

The traveling density waves referred to earlier may or may not be moving striations. Certainly one could be excused if in the first glow of understanding, he impetuously ascribed the title of moving striations to these traveling density waves; however, sober and mature consideration would make such a conclusion doubtful. The important point to remember here is that moving striations are an observable phenomena and not an ethereal speculation.

A mathematical approach by Watanabe and Oleson has been undertaken. In their approach they attempt to provide a theoretical basis for explaining moving striations. However, they assert that certain of their approximations may be in error if applied to actual striations(1). It is the intent hereafter to justify or refute these assumptions. In order to do this it is proposed to extend the linear treatment of Watanabe and Oleson to the non-linear case and thereby arrive at a more general solution. A search of the literature was instituted to garner the latest, most exact data available. Where these data were found to be either doubtful or non-existent, data were taken from current experiments being performed at the U. S. Naval Postgraduate School. The ultimate aim was to substitute these data in the pertinent mathematical equations and solve them by means of a digital computer. The solutions then could be compared with the results obtained experimentally and a reasonable conclusion could be reached as to the validity of the original premises.

Since we have assumed that we have a fluid filled tube, let us establish a coordinate system to describe the particles in our fluid. In this system the neutral particles are always fixed in space and only the positive and negative particles move with respect to our coordinate system.

If ρ is the density of any physical quantity, Q the rate per unit volume at which the quantity is being produced, and δV the volume occupied by the quantity; then,

$$\frac{d}{dt}(\rho \delta V) = Q \delta V$$

$$\text{or} \qquad \delta V \frac{d\rho}{dt} + \rho \frac{d \delta V}{dt} = Q \delta V$$

Let us define δV as a small volume of fluid and agree that δV always designates a volume element that moves with the fluid such that it contains the same fluid particles. In general the volume, δV, will then change with time. If we assume that δV is in the form of a rectangular box, then let us say that its dimensions are δx, δy, and δz, such that;

$$\delta V = \delta x \, \delta y \, \delta z$$

Now, the x-component of fluid velocity, v_x, may be different for the opposing faces of our box moving in the x-direction. If so, δx will change with time at a rate equal to the difference between these two velocities; thusly,

$$\frac{d}{dt}\delta x = \frac{\partial v_x}{\partial x}\delta x$$

and by a similar development we can say,

$$\frac{d}{dt}\delta y = \frac{\partial v_y}{\partial y} \qquad \text{and} \qquad \frac{d}{dt}\delta z = \frac{\partial v_z}{\partial z}\delta z$$

Therefore,

$$\frac{d\delta V}{dt} = \delta y\,\delta z\,\frac{d}{dt}\delta x + \delta x\,\delta z\,\frac{d}{dt}\delta y + \delta x\,\delta y\,\frac{d}{dt}\delta z$$

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta x\,\delta y\,\delta z$$

$$\frac{d\delta V}{dt} = \nabla\cdot\vec{v}\,\delta V$$

Inserting this in the equation above for $\frac{d\delta V}{dt}$ we have:

$$\delta V\frac{d\rho}{dt} + \rho\nabla\cdot\vec{v}\,\delta V = Q\,\delta V$$

dividing out δV,

$$\frac{d\rho}{dt} + \rho\nabla\cdot\vec{v} = Q$$

~~but~~ by definition,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

therefore,

$$\frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v} = Q$$

Which by a relation in vector analysis reduces to:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = Q$$

This of course is nothing more than the equation of continuity. A more rigorous derivation can usually be found in most texts on mechanics.*

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*See for instance K. R. Symon, Mechanics (Addison-Wesley Publishing Company, Inc., Cambridge, Massachusetts, 1953), p. 265 ff.

For the sake of simplicity we shall assume that all of the negative ions are electrons and that the positive ions are formed by the collision of electrons with neutral particles and by no other process--a reasonable assumption if we know that our gas is not an electro-negative gas(3). See fig. 1 for a simple graphical representation of this ionization phenomenon.

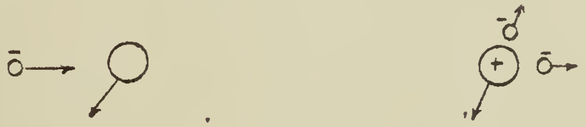

Fig 1 (Ionized Gases, von Engel)

If we place a charged particle in an electric field, the particle will move under the influence of the Coulomb law of attraction. It seems reasonable that if there are neutral particles in the field which are unaffected by Coulomb forces, the charged particles moving in the field will collide with the neutral ones. This obviously occurs. However, since the charged particle is still under the influence of the field, the particle will again tend to move after collision toward the electrode opposite in sign to the particle. The overall effect is to give the charged particle a motion generally in one direction (see fig. 2).

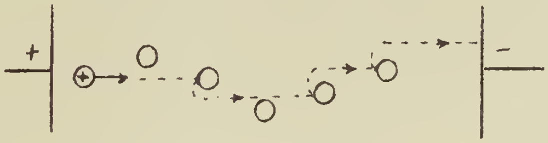

Fig. 2 (Ionized Gases, von Engel)

Since the Coulomb forces operate to move the particle, it would appear that the same law acting between oppositely charged particles would cause them to recombine. This does happen under proper conditions; i.e., high pressures (of the order of one atmosphere). However, at low pressures the collision frequency and collision losses are low and the relative

velocity of the charged particles is higher. Because the velocities are higher, collision without recombination takes place since the attracting field is not powerful enough to deflect the ions and make them move closely together over a large enough distance(4).

Let us now define:

n^- = negative ion density

n^+ = positive ion density

v^- = negative ion drift velocity

v^+ = positive ion drift velocity

z = number of ion pairs created per unit time per one, electron (constant if the temperature variations are very small(1))

then from the equation of continuity:

eqn (1)

$$\frac{\partial n^-}{\partial t} + \nabla\cdot(n^-\vec{v}^-) - zn^- = 0$$

$$\frac{\partial n^+}{\partial t} + \nabla\cdot(n^+\vec{v}^+) - zn^- = 0$$

Further consideration of a charged particle in an electric field will inevitably lead to the conclusion that the velocity of the ion is a function of the field. Experimental evidence shows that the velocity is proportional to the field to a first approximation, viz:

$$\vec{v} \sim E$$

(see figs. 3 and 4)

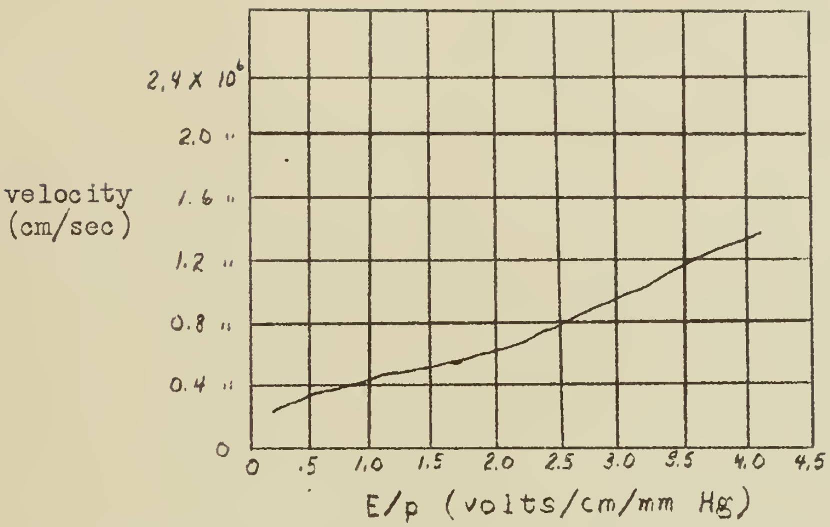

Figure 3. Drift velocity of electrons in argon as a function of E/p. R. A. Nielsen, Phys. Rev. 50, 950 (1936)

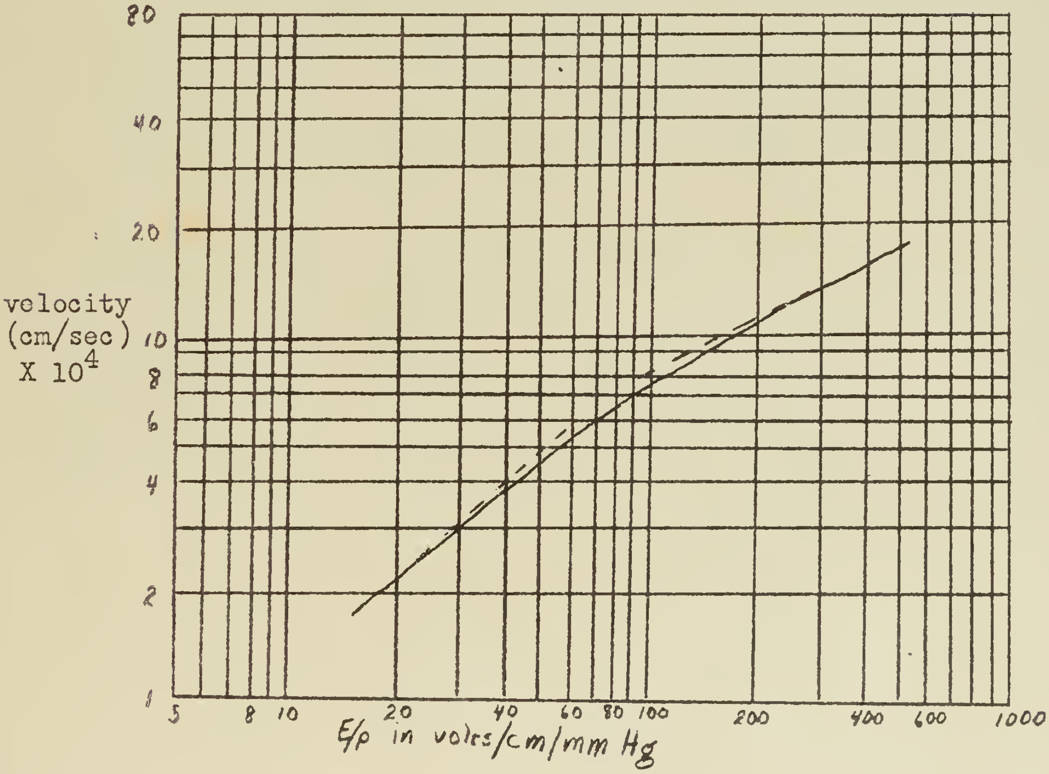

Figure 4. The drift velocity of Argon atomic ions in Argon as a function of E/p . J. A. Hornbeck, Phys. Rev. 84, 615 (1951)

The use of vector notation is justified when consideration is given to the fact that the motion of the ion is directed. We may therefore say:

$$\vec{V}_{FIELD} = KE$$

where K is a proportionality constant called the mobility--a quantity which will be more fully discussed later. Considering again figs. 3 and 4, it can be seen that the magnitude of the velocity of the electron is much greater than that of the positive ion; indeed, in certain instances the velocity of the electron can be shown to be a thousand times as great as that of the positive ion. This fascinating facet of the properties will be given more light later in the discussion of ambipolar diffusion.

Envisioning a cloud of ions (or electrons) in the tube, one is led to the inevitable conclusion that the charged particles may exist in varying densities. If this be the case, then obviously the ions will flow from regions of higher density to the regions of lower density by diffusion. This will of course contribute a component to the velocity at which the ions move, and we can say:

$$\vec{V}_{DIFFUSION} \sim -\nabla n$$

where ∇n is the concentration gradient with the negative sign indicating that the motion occurs in the direction of decreasing concentration. Therefore,

$$\vec{V}_{DIFFUSION} = -\frac{D}{n}\nabla n$$

where D is a proportionality constant called the diffusion coefficient for the gas. A more adequate treatment of the derivation of this last equation can be found in some detail in most texts dealing with the subject

of diffusion.* Combining these contributions to velocity we have,

eqn (2) $\vec{V} = KE - \frac{D}{n}\nabla n$

The electric field under which the particles act is composed of two parts, the impressed electric field and the electric field generated by the space charges; such that,

$$E = E_0 + E'$$

where E_0 is the impressed electric field and will be considered constant, and E' is the electric field generated by the space charges and depends on the number of electrons and ions present; i.e., the density. Therefore we can say (from Poisson's Equation):

$$\nabla\cdot E = \nabla\cdot(E_0 + E') = \nabla\cdot E' = 4\pi e(n^+ - n^-)$$

where e is the electronic charge.

It is proposed to solve equations (1) by assuming that the actual solution is a superposition of a steady uniform solution $n_0^\pm$ plus a non-steady, nonuniform disturbance $\nu^\pm$. In order to linearize the equation Watanable and Oleson assumed that the nonsteady nonuniform disturbance was small compared to the uniform, steady disturbance, which is not in agreement with experimental results.(5) (6) In the following development no such assumption will be made--to the contrary, we shall assume that at some particular time the orders of magnitude of the steady, uniform and

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*See for instance L. B. Loeb, Basic Processes of Gaseous Electronics (University of California Press, Berkeley and Los Angeles, 1955), p.188 ff.

nonsteady, nonuniform parts may be approximately equal. This is not to imply that they are always equal, but rather that they could be equal at some particular time. The word "uniform" as used here means that the quantities do not depend on the x-coordinate taken along the axis of the tube(1). In keeping with our proposed solutions let us now write:

$$\text{eqn (3)} \qquad \begin{aligned} n^+ &= n_0^+ + \nu^+ \\ n^- &= n_0^- + \nu^- \end{aligned}$$

We define $\mathcal{E}$ as the field generated by the steady, uniform charge densities $n_0^\pm$.

Substituting the value given for $\vec{V}$ in equation (2) into equation (1):

$$\frac{\partial n^-}{\partial t} + \nabla \cdot \left[n^- \left(-K^- E - \left\{ \frac{D^-}{n^-} \nabla n^- \right\} \right) \right] - z n^- = 0$$

$$\frac{\partial n^+}{\partial t} + \nabla \cdot \left[n^+ \left(K^+ E - \left\{ \frac{D^+}{n^+} \nabla n^+ \right\} \right) \right] - z n^- = 0$$

clearing the parentheses:

$$\frac{\partial n^-}{\partial t} + \nabla \cdot \left[-n^- K^- E - \left\{ D^- \nabla n^- \right\} \right] - z n^- = 0$$

$$\frac{\partial n^+}{\partial t} + \nabla \cdot \left[n^+ K^+ E - \left\{ D^+ \nabla n^+ \right\} \right] - z n^- = 0$$

Substituting equation (3) in the preceding equations:

$$\frac{\partial n_0^-}{\partial t} + \frac{\partial \nu^-}{\partial t} + \nabla \cdot \left[-n_0^- K^- E - \nu^- K^- E - D^- \nabla n_0^- - D^- \nabla \nu^- \right] - z n_0^- - z \nu^- = 0$$

$$\frac{\partial n_0^+}{\partial t} + \frac{\partial \nu^+}{\partial t} + \nabla \cdot \left[n_0^+ K^+ E + \nu^+ K^+ E - D^+ \nabla n_0^+ - D^+ \nabla \nu^+ \right] - z n_0^- - z \nu^- = 0$$

Directing our attention to 'the terms containing $n_0^\pm$ and recalling that by definition these are constant; therefore,

$$\frac{\partial n_0^\pm}{\partial t} = 0 \quad \text{and} \quad D^\pm \nabla n_0^\pm = 0$$

which leaves,

$$\frac{\partial \nu^-}{\partial t} + \nabla\cdot\left[-n_0^- K^- E - \nu^- K^- E - D^- \nabla \nu^-\right] - z n_0^- - z\nu^- = 0$$

$$\frac{\partial \nu^+}{\partial t} + \nabla\cdot\left[n_0^+ K^+ E + \nu^+ K^+ E - D^+ \nabla \nu^+\right] - z n_0^- - z\nu^- = 0$$

but we have stipulated that $E = E_0 + E'$; it then follows:

$$\frac{\partial \nu^-}{\partial t} + \nabla\cdot\left[-n_0^- K^- E_0 - n_0^- K^- E' - \nu^- K^- E_0 - \nu^- K^- E' - D^- \nabla \nu^-\right] - z n_0^- - z\nu^- = 0$$

$$\frac{\partial \nu^+}{\partial t} + \nabla\cdot\left[n_0^+ K^+ E_0 + n_0^+ K^+ E' + \nu^+ K^+ E_0 + \nu^+ K^+ E' - D^+ \nabla \nu^+\right] - z n_0^- - z\nu^- = 0$$

Again we refer to the fact that E_0 is a constant, consequently,

$$\nabla\cdot E_0 = 0$$

Omitting the zero terms, our equations become:

$$\frac{\partial \nu^-}{\partial t} + \nabla\cdot\left[-n_0^- K^- E' - \nu^- K^- E_0 - \nu^- K^- E' - D^- \nabla \nu^-\right] - z n_0^- - z\nu^- = 0$$

$$\frac{\partial \nu^+}{\partial t} + \nabla\cdot\left[n_0^+ K^+ E' + \nu^+ K^+ E_0 + \nu^+ K^+ E' - D^+ \nabla \nu^+\right] - z n_0^- - z\nu^- = 0$$

We have defined $\mathcal{E}$ as the field generated by the uniform, steady charge density $n_0^\pm$. Let us now define $\vec{e}$ as the field due to the nonuniform, nonsteady disturbance $\nu^\pm$; such that,

$$E' = \mathcal{E} + \vec{e}$$

Inserting this relation into the equations above:

$$\frac{\partial \nu^-}{\partial t} + \nabla\cdot\left[-n_0^- K^- \mathcal{E} - n_0^- K^- \vec{e} - \nu^- K^- E_0 - \nu^- K^- \mathcal{E} - \nu^- K^- \vec{e} - D^- \nabla \nu^-\right] - z n_0^- - z\nu^- = 0$$

$$\frac{\partial \nu^+}{\partial t} + \nabla\cdot\left[n_0^+ K^+ \mathcal{E} + n_0^+ K^+ \vec{e} + \nu^+ K^+ E_0 + \nu^+ K^+ \mathcal{E} + \nu^+ K^+ \vec{e} - D^+ \nabla \nu^+\right] - z n_0^- - z\nu^- = 0$$

Removing brackets,

$$\frac{\partial \nu^-}{\partial t} - n_0^- K^- \nabla\cdot\mathcal{E} - n_0^- K^- \nabla\cdot\vec{e} - K^- E_0 \nabla\cdot\nu^- - K^-\nu^-\nabla\cdot\mathcal{E} - K^-\mathcal{E}\cdot\nabla\nu^- - K^-\nu^-\nabla\cdot\vec{e} - K^-\vec{e}\cdot\nabla\nu^-$$
$$- D^-\nabla^2\nu^- - zn_0^- - z\nu^- = 0$$

$$\frac{\partial \nu^+}{\partial t} + n_0^+ K^+ \nabla\cdot\mathcal{E} + n_0^+ K^+ \nabla\cdot\vec{e} + K^+ E_0 \nabla\cdot\nu^+ + K^+\nu^+\nabla\cdot\mathcal{E} + K^+\mathcal{E}\cdot\nabla\nu^+ + K^+\nu^+\nabla\cdot\vec{e} + K^+\vec{e}\cdot\nabla\nu^+$$
$$- D^+\nabla^2\nu^+ - zn_0^- - z\nu^- = 0$$

From Poisson's Equation:

$$\nabla\cdot\mathcal{E} = 4\pi e(n_0^+ - n_0^-) \quad \text{and} \quad \nabla\cdot\vec{e} = 4\pi e(\nu^+ - \nu^-)$$

where e is the charge on the electron. Recalling that our region of interest is near the axis of the tube removed from the ends and assuming cylindrical symmetry, we find:

$$\nabla\nu^\pm = \frac{\partial\nu^\pm}{\partial x} \quad \text{and} \quad \vec{e} = \vec{e}_x$$

With these conditions in mind let us rewrite the equations:

$$\frac{\partial \nu^-}{\partial t} - n_0^- K^-[4\pi e(n_0^+ - n_0^-)] - n_0^- K^-[4\pi e(\nu^+ - \nu^-)] - K^- E_0\frac{\partial\nu^-}{\partial x} - K^-\nu^-[4\pi e(n_0^+ - n_0^-)] - K^-\mathcal{E}\cdot\frac{\partial\nu^-}{\partial x}$$
$$- K^-\nu^-[4\pi e(\nu^+ - \nu^-)] - K^-\vec{e}_x\frac{\partial\nu^-}{\partial x} - D^-\frac{\partial^2\nu^-}{\partial x^2} - zn_0^- - z\nu^- = 0$$

$$\frac{\partial \nu^+}{\partial t} + n_0^+ K^+[4\pi e(n_0^+ - n_0^-)] + n_0^+ K^+[4\pi e(\nu^+ - \nu^-)] + K^+ E_0\frac{\partial\nu^+}{\partial x} + K^+\nu^+[4\pi e(n_0^+ - n_0^-)] + K^+\mathcal{E}\cdot\frac{\partial\nu^+}{\partial x}$$
$$+ K^+\nu^+[4\pi e(\nu^+ - \nu^-)] + K^+\vec{e}_x\frac{\partial\nu^+}{\partial x} - D^+\frac{\partial^2\nu^+}{\partial x^2} - zn_0^- - z\nu^- = 0$$

Inasmuch as our interest in the field is in the central region along the axis of the tube, we note that the field $\mathcal{E}$ (or more particularly $\mathcal{E}_x$) will be zero because of the steady, uniformity of the charge density $n_0^\pm$ causing it. Further, in this region n_0^+ is approximately equal to n_0^-.

eqn (4)
$$\frac{\partial \nu^-}{\partial t} - D^-\frac{\partial^2\nu^-}{\partial x^2} - K^- E_0\frac{\partial\nu^-}{\partial x} + \gamma_1\nu^- - \gamma_2\nu^+ - K^-\vec{e}_x\frac{\partial\nu^-}{\partial x} - \gamma_5(\nu^+ - \nu^-)\nu^- - zn_0^- = 0$$
$$\frac{\partial \nu^+}{\partial t} - D^+\frac{\partial^2\nu^+}{\partial x^2} + K^+ E_0\frac{\partial\nu^+}{\partial x} + \gamma_3\nu^+ - \gamma_4\nu^- + K^+\vec{e}_x\frac{\partial\nu^+}{\partial x} + \gamma_6(\nu^+ - \nu^-)\nu^+ - zn_0^- = 0$$

where,

$$\gamma_1 = 4\pi e K^-(2n_0^- - n_0^+) - z \qquad \gamma_4 = 4\pi e K^+ n_0^+ + z$$

$$\gamma_2 = 4\pi e K^- n_0^- \qquad \gamma_5 = 4\pi e K^-$$

$$\gamma_3 = 4\pi e K^+(2n_0^+ - n_0^-) \qquad \gamma_6 = 4\pi e K^+$$

For comparison purposes the terms,

$$-K^-\vec{e}_x \frac{\partial v^-}{\partial x} - \gamma_3 (v^+ - v^-)v^- - z n_0^-$$

$$+K^+\vec{e}_x \frac{\partial v^+}{\partial x} + \gamma_6 (v^+ - v^-)v^+ - z n_0^-$$

which appear in equations (4) do not appear in the original paper by Watanabe and Oleson. These, then, are the terms dropped by Watanabe and Oleson as a result of the mathematical expediency employed by them to linearize their equations, and as a result of disregarding certain higher order terms.$^{(1)}$

Equations (4) are the ones for which constants will be computed for a particular set of conditions; e.g., Argon gas, 12 mm Hg pressure, etc.

CHAPTER II

EVALUATION OF CONSTANTS

Reiterating our original contention, we propose to apply equations (4) to an experimental situation. In order to do this certain constant terms must be evaluated. Other constants are well known but will be repeated here in order that our results may be identically reproduced. In this latter category of constants we find:

$e = 4.8 \times 10^{-10}$ e.s.u.

$k = 1.38 \times 10^{-16}$ ergs/ °K

$\pi = 3.1416$

In the former class of constants there will be a need to know the values of K^+, K^-, D^+, D^-, E_0, z, n_0^-, and n_0^+. Rather than list them arbitrarily, it seems more appropriate to derive them individually in order to indicate their reliability.

The particular experimental situation to which we shall apply equations (4) is as follows:

> A tube 0.5 cm in radius containing pure Argon at a pressure of 12 mm Hg with currents of from 20ma to 100ma in our gaseous conductor(6).

With these conditions the value of the applied electric field can be found. From fig. 5[*] it can be seen that for a pressure-radius product of 6 we have in our case an $\frac{E_0}{p}$ ratio of approximately 0.5. At a pressure of 12 mm Hg this will yield an E_0 of 6 volts/cm. The value of E_0 which we obtained for our case yields a value of 2×10^{-2} statvolts when converted to e.s.u.

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[*] For a description of the method of obtaining these curves refer to the source cited in connection with figure 5.

From probe measurements in the plasma within the tube it has been experimentally determined that (6),

$$n_o^- \doteq n_o^+ \doteq 7 \text{ X } 10^{10} \text{ ions/cm}^3$$

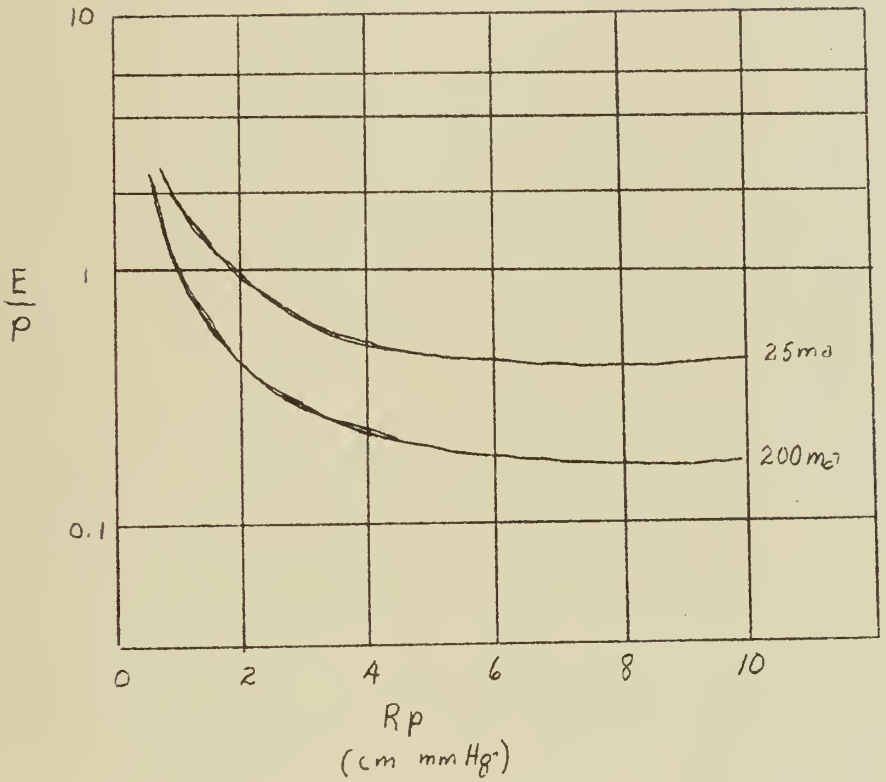

Figure 5
(Ionized Gases, von Engel)

It appears rather obvious that a relation exists between mobility, diffusion, and the number of ion pairs created per unit time per one electron. Since the latter two terms have been defined, it remains only to define the first term--mobility. We have rather arbitrarily stated that mobility is the proportionality constant which relates the drift velocity to the field. However, it seems more important to have a "feeling" for the phenomenon of mobility than to have a pedantic knowledge of the word itself. Consider again a charged particle moving under the

influence of a field and within a region occupied by the neutral particles of a gas (fig. 2). We can say, then, that mobility is the drift velocity of an ion moving in a unit field.*

In order to utilize the experimental data available(6), it is necessary to determine values of mobility for the positive ions and the electrons. In dealing with positive ion mobility it was found that there is a difference in mobilities depending on whether the ions present are "atomic" ions or "molecular" ions. It has been shown that when pressures of the order of 6 mm Hg are encountered in Argon, A_2^+ and A^+ ions are found

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*In searching for a best value for mobility the phrase "reduced mobility" was often encountered. Reduced mobility is best described as(7):

$$k = \frac{(v)(p)(273)}{(E)(760)(T)}$$

where p is the gas pressure and T is the gas temperature in °K. Many results are given in terms of reduced pressure, p_o, (see for example fig. 4). This reduced pressure is defined as(8):

$$p_o = \frac{273p}{T}$$

Very often in studying graphically presented results one observes that experimenters and theoreticians will plot ion mobility versus field to pressure ratio (i.e., $\frac{E}{p}$) --where the pressure may be reduced pressure or that of the gas; e.g., figs. 3 and 4. It can be shown(2) that p is inversely proportional to the mean free path, thusly:

$$p \sim \frac{1}{\lambda}$$

If this be so, then:

$$\frac{E}{p} \sim E\lambda$$

where $E\lambda$ is an expression for the energy acquired in a m.f.p. This gives the researcher a more vivid and accurate picture of what is happening.

to be present in the ratio of 0.5(3). The mobilities of these two ions were found to be 2.6 and 1.6 $\frac{\text{cm/sec}}{\text{volt/cm}}$ respectively(9). Assuming that the two types of ions are distributed in the ratio indicated throughout the positive column, a weighted average mobility was found to be:

$$K^{+} = 4.04 \times 10^{4} \text{ cm}^{2}/\text{stv-sec}$$

If the two types of ions are assumed to be present in equal concentrations, then:

$$K^{+} = 4.38 \times 10^{4} \text{ cm}^{2}/\text{stv-sec}$$

As can be seen, the orders of magnitude are the same or at least near enough that either term could be used with some degree of accuracy.

From fig. 3 the drift velocity of an electron was determined to be 0.36×10^{6} cm/sec. To a first approximation,

$$K^{-} = \frac{v}{E} = \frac{0.36 \times 10^{6}}{2 \times 10^{-2}} = 1.8 \times 10^{7} \text{ cm}^{2}/\text{stv-sec}$$

The diffusion coefficients, D^{+} and D^{-}, are most easily found by employing the mobility. The relationship between mobility and diffusion is derived in numerous texts on the subject(2). The relation is:

$$\frac{K}{D} = \frac{e}{kT}$$

where e is the electronic charge, k is Boltzmann's constant, and T is the ion temperature in °K. The temperature of the ions is approximately equal to the temperature of the gas(10). From this equation, then, the following values were determined:

$$D^{+} = 3.48 \text{ cm}^{2}/\text{sec}$$

$$D^{-} = 6.46 \times 10^{4} \text{ cm}^{2}/\text{sec}$$

In order to compute the value of z it is necessary to employ ambi-

polar diffusion in the positive column results from the previously shown fact that the diffusion coefficient for electrons is greater than for ions. Assume that as an original condition there exists near the axis of the positive column an almost equal number of electrons and positive ions. After a short interval the electrons will tend to diffuse toward the walls of the tube at a rate greater than the positive ions. This will cause a negative charge to be set up on the walls of the tube. The preponderance of positive ions remaining near the axis of the tube (due to their slower diffusion) causes an electric field to be established between the axis of the tube and the wall. This field in turn will cause a slowing down of the electrons moving toward the wall and a speeding up of the ions. It happens that wall recombination is preponderant at the pressure we have specified. This is borne out by the absence of recombination spectra indicating that volume recombination is not important here. In an equilibrium state the losses due to recombination are equal to the rates of production. The electric field across the positive column is such that electrons arriving from the faraday dark space are accelerated to produce enough electrons and ions per unit length of the tube to compensate for losses, thus establishing equilibrium(2). Specifically, in our case we have restricted our consideration to the region near the axis of the tube so that radial variations need not be considered. The point to be remembered here is that under the action of the radial electric field, an average velocity of diffusion is soon reached such that,

$$v_r^- = v_r^+ = v_{r_{av}}, \quad \frac{dn_o^-}{dr} = \frac{dn_o^+}{dr}$$

which leads to the fact that the total ambipolar current is zero. As we have already seen,

$$n_o^+ \doteq n_o^-$$

Also, we have previously implied (page 13) that near the axis of the tube $\vec{e}_r = 0$ and $\mathcal{E}_x = 0$. Using the expressions for velocity and eliminating the field between them, it can be shown that$^{(2)}$:

$$D_a = \frac{D^+K^- + D^-K^+}{K^+ + K^-}$$

where D_a is defined as the ambipolar diffusion coefficient. Further, since K^- is much greater than K^+ and T_e is much greater than T_p (where T_e and T_p are the electron and ion temperatures respectively), it can be shown that:

$$D_a = D^+ \frac{T_e}{T_p} = D^- \frac{K^+}{K^-} - K^+ \frac{kT_e}{e}$$

Employing known data $^{(6)}$ ($T_e = 12,550$ °K and $T_p = 300$ °K) in the approximate formula for D_a yields:

$$D_a = 145 \text{ cm}^2/\text{sec}$$

From the ambipolar solution to our problem it can be shown that:

$$z = \frac{(2.405)^2}{(R)^2} D_a$$

where R is the tube radius. Inserting the value for D_a computed above and the radius of our tube--0.5 cm$^{(6)}$ for the conditions we have chosen, we find:

$$z = 3.35 \times 10^3 \text{ ion pairs/electron-sec}$$

Inserting the constants just computed into equations (4) yields:

$$\frac{\partial \nu^-}{\partial t} - 6.46 \times 10^4 \frac{\partial^2 \nu^-}{\partial x^2} - 3.6 \times 10^5 \frac{\partial \nu^-}{\partial x} + 7.61 \times 10^9 (\nu^- - \nu^+) - 1.8 \times 10^7 \vec{e}_x \frac{\partial \nu^-}{\partial x} - .1085 (\nu^+ - \nu^-) \nu^- - 2.345 \times 10^{14} = 0$$

$$\frac{\partial \nu^+}{\partial t} - 3.46 \frac{\partial^2 \nu^+}{\partial x^2} + 808 \frac{\partial \nu^+}{\partial x} + 1.71 \times 10^7 (\nu^+ - \nu^-) + 4.04 \times 10^4 \vec{e}_x \frac{\partial \nu^+}{\partial x} + 2.44 \times 10^{-4} (\nu^+ - \nu^-) \nu^+ - 2.345 \times 10^{14} = 0$$

Little accuracy is sacrificed by disregarding those values of constants which are dominated by the exceedingly large values of other constants which are additive.

In order to solve this rather formidable differential equation, little error is involved if the partial derivatives are approximated by total derivatives and minutely small increments are taken to be used in a digital computer. This may be more clearly seen later where the above equations are set up for programming into the computer. Once the equations are written in a "useful form" for the computer, we find we must have some starting point; i.e., some initial values for ν^t, x, e_{x_i}, etc. at some time, t. Since the approach we have taken appears to be a rather unique one, there were no readily available precedents by which we could be guided. Harking back to the original intent of this thesis, the answer suggested itself. Since Watanabe and Oleson(1) assumed a solution to their linear equation, it seemed reasonable to take their evaluation and apply it to our problem. If this satisfied our equations, then credence is lent to the original hypothesis thereby extending the theoretical contention to actual conditions. Watanabe and Oleson(1) assumed linear solutions:

$$\nu^- = a^+ e^{ikx-i\omega t}$$

$$\nu^+ = a^- e^{ikx-i\omega t}$$

where a^+ and a^- are complex amplitudes, k is the wave number, and ω is the angular frequency. Thus, another problem arose--the feeding of a complex number into a digital computer. It is not impossible to execute such a program, but in view of the physical construction of the computer, it would be necessary to sacrifice some of the accuracy of the original program in

order to do so. However, since our solution consists of a real part and an imaginary part; that is to say, a real solution and an imaginary solution, either is a solution to our problem. So, we will program only the real solution. This reduces our solution to:

$$\nu^- = a^- \cos(kx-\omega t)$$

$$\nu^+ = a^+ \cos(kx-\omega t)$$

Letting the time arbitrarily be zero initially these reduce to:

$$\nu_0^- = a^- \cos kx_0$$

$$\nu_0^+ = a^+ \cos kx_0$$

Again we have introduced terms which must be evaluated. Directing our attention to k, the wave number, and referring to the work of Watanabe and Oleson(1), we observe that we must assume their linear counterparts of our equations (4). This results in no inconsistencies when the restrictions placed on our equations are considered. Watanabe and Oleson's equations are:

$$\frac{\partial \nu^-}{\partial t} - K^- E_0\left(\frac{\partial \nu^-}{\partial x}\right) - D^- \frac{\partial^2 \nu^-}{\partial x^2} + \gamma_1 \nu^- - \gamma_2 \nu^+ = 0$$

$$\frac{\partial \nu^+}{\partial t} + K^+ E_0\left(\frac{\partial \nu^+}{\partial x}\right) - D^+ \frac{\partial^2 \nu^+}{\partial x^2} + \gamma_3 \nu^+ - \gamma_4 \nu^- = 0$$

where γ_1, γ_2, γ_3, and γ_4 are the same as the ones previously defined.

Substituting

$$\nu^- = a^- e^{ikx-\omega t}$$

$$\nu^+ = a^+ e^{ikx-\omega t}$$

into Watanabe and Oleson's equations and performing the indicated operations will give:

$$(-i\omega - iK^-E_0k + D^-k^2 + \gamma_1)a^- - \gamma_2 a^+ = 0$$

$$-\gamma_4 a^- + (-i\omega + iK^+E_0k + D^+k^2 + \gamma_3)a^+ = 0$$

In order for two homogeneous equations to have a solution the determinant of coefficients must be zero.

$$\begin{vmatrix} -i\omega - iK^{-}E_0k + D^{-}k^2 + \gamma_1 & -\gamma_2 \\ -\gamma_4 & -i\omega + iK^{+}E_0k + D^{+}k^2 + \gamma_3 \end{vmatrix} = 0$$

Upon solving this determinant we have,

$$\left[-i\omega - iK^{-}E_0k + D^{-}k^2 + \gamma_1\right]\left[-i\omega + iK^{+}E_0k + D^{+}k^2\ \gamma_3\right] - \gamma_2\gamma_4 = 0$$

Recalling that if a complex equation can be written:

$$R + iI = 0$$

where R is the real part and iI the imaginary part, then R = 0 and iI = 0 for the equation to be satisfied.

More particularly for our case,

for R = 0,

$$-\omega^2 + \omega K^{+}E_0k - \omega K^{-}E_0k + K^{-}K^{+}E_0^2k^2 + (D^{-}k^2 + \gamma_3) - \gamma_2\gamma_4 = 0$$

for I = 0 (since $i \neq 0$),

$$-\omega\left[D^{+}k^2 + \gamma_3\right] - K^{-}E_0k\left[D^{+}k^2 + \gamma_3\right] - \omega\left[D^{-}k^2 + \gamma_1\right] + K^{+}E_0k\left[D^{-}k^2 + \gamma_1\right] = 0$$

Solving this last equation for ω:

$$\omega = \frac{kE_0\left[k^2(K^{+}D^{-} - K^{-}D^{+}) - (K^{-}\gamma_3 - K^{+}\gamma_1)\right]}{\left[(D^{+} + D^{-})k^2 + (\gamma_1 + \gamma_3)\right]}$$

Eliminating ω between the two equations yields,

eqn (6)
$$(D^{-}k^2 + \gamma_1)(D^{+}k^2 + \gamma_3) - \gamma_2\gamma_4 + \frac{k^2E_0^2(K^{+} + K^{-})^2(D^{+}k^2 + \gamma_3)(D^{-}k^2 + \gamma_1)}{\left[(D^{+}k^2 + \gamma_3) + (D^{-}k^2 + \gamma_1)\right]^2} = 0$$

Introducing new variables,

eqn (7) $\xi = D^{-}k^2 + \gamma_1$ and $\eta = D^{+}k^2 + \gamma_3$

Solving for k^2 in the first of equations (7),

$$k^2 = \frac{\xi - \gamma_1}{D^-}$$

Substituting this value for k^2 in the second equation (7),

$$\eta = \gamma_3 + \frac{D^+}{D^-}(\xi - \gamma_1)$$

Taking cognizance of the fact that $D^- \gg D^+$, we can say:

$$\eta \doteq \gamma_3$$

Referring to the original expression for η,

$$\eta = D^+k^2 + \gamma_3$$

and considering the preceding approximation, we observe:

$$D^+k^2 \doteq 0$$

Since D^+ is not zero (see page 18), this approximation can be true if and only if,

$$k^2 \doteq 0$$

Inserting the approximate values of η and ξ from equation (7), et seq, into equation (6) and observing that $k^2 \doteq 0$,

$$\xi \doteq \frac{\gamma_2 \gamma_4}{\gamma_3} \doteq \gamma_1$$

If we map equation (6) in the $\xi - \eta$ plane, we have approximated very closely a point on the extremely complicated curve represented by equation (6); i.e., the point where $k^2 \doteq 0$. It can be shown(1) from an elementary geometrical relationship that knowing a point and the slope of a curve at the point, the equation of the tangent can be written. In our case,

eqn (8) $$\eta = \gamma_3 - \left[\frac{(1+g)^2}{g(1+g)^2 + \frac{gh}{\gamma_3}}\right](\xi - g\gamma_3)$$

where, $g = \frac{\gamma_2 \gamma_4}{\gamma_3^2}$ and $h = \frac{(K^+ + K^-)^2 E_o^2}{D^+}$

Substituting the values of η and ξ as shown in equations (7) for the appropriate terms in equation (8), and noting that:

$$z \ll 4\pi e K^+ n$$

we solve for k^2:

$$k^2 = \frac{(K^+ + K^-)z}{K^+ D^- + K^- D^+ + \frac{K^+ E_o^2 K^-}{4\pi n e}}$$

Recalling the proposed linear solution, we direct our attention to the complex amplitudes and write:

$$\frac{a^+}{a^-} = Ae^{-i\phi} \quad \text{(with A and } \phi \text{ real)}$$

which can be written:

$$\frac{a^+}{a^-} = A(\cos\phi - i\sin\phi)$$

but as has been shown,

$$(-i\omega - iK^- E_o k + D^- k^2 + \gamma_1)a^- - \gamma_2 a^+ = 0$$

From these last two equations we can say,

$$\frac{a^+}{a^-} = A\cos\phi - iA\sin\phi = \frac{D^- k^2 + \gamma_1 - i(\omega + K^- E_o k)}{\gamma_2}$$

Equating reals and imaginaries,

$$A\cos\phi = \frac{D^- k^2 + \gamma_1}{\gamma_2} \quad \text{and} \quad A\sin\phi = \frac{\omega + K^- E_o k}{\gamma_2}$$

Solving for A,

$$A = \frac{D^- k^2 + \gamma_1}{\gamma_2 \cos\phi} = \frac{D^- k^2 + \gamma_1}{\gamma_2} \sqrt{1 + \tan^2\phi}$$

From which,

$$\tan \phi = \frac{\omega + K^{-}E_{o}k}{D^{-}k^{2} + \gamma_{1}}$$

Inserting,

$$\omega = \frac{kE_{o}\left[k^{2}(K^{+}D^{-} - K^{-}D^{+}) - (K^{-}\gamma_{3} - K^{+}\gamma_{1})\right]}{\left[(D^{+} + D^{-})k^{2} + (\gamma_{1} + \gamma_{3})\right]} \quad \text{and}$$

Simplifying yields,

$$\text{eqn (9)} \quad \tan \phi = \frac{(K^{+} + K^{-})kE_{o}}{\left[(D^{+} + D^{-})k^{2} + (\gamma_{1} + \gamma_{3})\right]}$$

It can be shown(1) that tan ϕ has one maximum between 0 and $\pi/2$.

Taking the derivative of tan ϕ with respect to k and equating to zero we find at this point:

$$\text{eqn (10)} \quad k^{2} = \frac{\gamma_{3} + \gamma_{1}}{\left[2E_{o}(K^{+} + K^{-})\right]\ (D^{+} + D^{-})^{2}}$$

Squaring equation (9) and inserting equation (10) gives:

$$(\tan\phi)^{2}_{max} = \frac{(K^{+} + K^{-})^{2}E_{o}}{\left[4(D^{+} + D^{-})(\gamma_{1} + \gamma_{3})\right]}$$

Calculating this value of ϕ from the known values of the constants in the preceding equation, we used this as the maximum value of a phase factor with which to enter the computer along with our previously described boundary conditions. The value of ϕ obtained as described was found to be 1°54'.

CHAPTER III

ASSUMPTIONS AND CRITIQUE OF ASSUMPTIONS--
SOME DIFFICULTIES ENCOUNTERED

The experimental data yield a Langmuir plot which is a straight line over two cycles leading to an assumed Maxwellian distribution for electrons(6). The assumption that the ions have a Maxwellian distribution stems from the fact that at several millimeters pressure, the temperature of the ions is approximately equal to that of the gas molecules(10). The assumption that we have radial ambipolar diffusion under our stated conditions within the tube is based on the results of Karge, Hooks, and Oleson(6). In this work they state that the instantaneous field intensity is due to a very small surplus of positive ions or electrons. Their reasoning for this is very amply stated in their work. Since we can say that $n^- \doteq n^+$, it follows that we can say the ambipolar current at the walls is approximately zero. Further,

$$\frac{dn^-}{dt} = \frac{dn^+}{dt} \quad \text{and} \quad v_r^- = v_r^+$$

where we define v_r^- and v_r^+ as the negative (i.e., electron) ion and positive ion radial drift velocities respectively.

In our preceding work and the subsequent computer work it was assumed that z was a constant. If z is also evaluated from the relation(4):

$$\text{eqn (11)} \quad z \doteq 9 \text{ X } 10^7 \, a p e^{-} \frac{\mathcal{E}_i}{\mathcal{E}_m}\left(\frac{kT}{e}\right)^{\frac{1}{2}} V_i$$

using the temperatures observed by Karge, Hooks, and Oleson(6), it is found that z varies widely with temperature as shown below:

T_e	z
16,440	$1.89 \text{ X } 10^5$
14,660	$5.3 \text{ X } 10^4$
8,200	2.22

Farris(11) notes a variation in z based on temperatures (the data being taken from Pupp's work). When this information is applied by the use of our equation (11), the following results are obtained:

T_e	z
28,000	2.7×10^7
11,520	1.63×10^3

It should be noted that the validity of the equation for z depends on having Maxwellian distribution of electrons which we have assumed.

Farris(11) has recently published a short paper on a theory of moving striations based on the diffusion theory. His basic assumption is quoted:

> If it is now assumed that the small difference between the ion and electron concentrations has the same effect axially as radially, and that mean values of mobilities and diffusion coefficients may be used, the theory of a stationary positive column may be readily generalized to include changes of concentration axially and with time. Eliminating the current and field from the usual conservation and current density equation gives: $S = dn/dt - D_a\ {}^2n$, where S is the rate of production of ions per unit volume per second and D_a the Schottky ambipolar diffusion coefficient given by: $D_a = kT_e K_+/e$.

The major assumption different from our assumption is that the axial drift velocities of the electrons and positive ions are assumed equal within the positive column. Using our equations (1) and eliminating the field we obtain: $n\vec{v} = D_a \nabla n$. Substituting this in our equation (1) we get Farris' equation (1). However, we question the validity of the assumption that the axial drift velocities of all ions in the positive column are equal. The interesting point is the variation of z with electron temperature.

From these considerations(6) we observe that there is a fallacy in our assumption that z is a constant. True, we imposed the condition that

this was valid only for small temperature variations. But at the time we imagined this variation could easily be on the order of 100 °K. Now, it appears that this is not so. Certainly this is a field for further inquiry. How this deviation from our original assumption affects our problem will be shown in the next chapter. We have taken zn_o^- to be a constant of the value of 2.345×10^{14}.

It seems noteworthy to point out that there is a phase difference between the maximum degree of ionization and the maximum degree of excitation which has been observed in some rare gases,* however this was not observed by Karge, Hooks, and Oleson(6). As a point in fact, they observed that a maximum in light intensity was accompanied by a maximum T_e, and a minimum in light intensity was accompanied by a minimum T_e. This would suggest that there is no phase difference in the case of Argon ions in Argon gas at 12 mm Hg pressure. If we can assume that equation (11) holds and further assume that the maximum light intensity is also evidence for maximum excitation, then we can say indeed there is no phase difference between the phenomena.

The temperature dependence of mobilities and diffusion coefficients was ignored in this thesis. Since the evaluated constants in equation (4) were taken to be average values(6), it is suspected that temperature dependence is of little significance.

The relationship between mobility and diffusion used in this thesis, $\frac{K}{D} = \frac{e}{kT}$, (page 18) may be a tenuous one. Von Engel(4) states that the

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*Private communication with N. L. Oleson, who with A. W. Cooper, has found that maximum light intensity and maximum electron temperature are not coincident in neon at several mm pressure.

the equation is not applicable to electrons for at least two reasons; electrons drift with velocities which are not proportional to the applied field and their mean free path depends on their speed. In defense of our use of this equation we relied on experimental data (see fig. 3) which shows that for our conditions the velocity is approximately proportional to the applied field. The $\frac{E}{p}_{\max}$ obtained by Karge, Hooks, and Oleson was 6 volts per cm per mm Hg. This figure is well within the range of fields wherein experimental data reveals that velocity is proportional to field (fig. 3). No attempt has been made to justify or refute the m.f.p. difficulty. Von Engel(4) further states that the equation holds in the case of ions which move and diffuse in moderately strong fields: in strong fields the velocity is proportional to the square root of the field. He further states that it can be assumed that the equation holds for weak fields. That our conditions fall within the purview of Von Engel's statements can be seen from figure 4, wherein the velocity of the ions is shown to be proportional to the field.

Graham and Ruhlig(12) obtained a value of 2×10^{6} cm^2/sec for the diffusion coefficient of the electron (D^-) by using Droyvesteyn distribution in conjunction with the Boltzmann transport equation. This value was calculated at the same conditions specified for our case. As can be seen from page 18 we calculated a value of 6.46×10^{4} cm^2/sec. No attempt was made to resolve this difference. However, as will be shown later, this difference may be rather easily investigated by use of the program we propose by merely varying the coefficients of certain terms.

CHAPTER IV

PROGRAMMING

In order to facilitate an understanding of the computer program we will set down a brief mathematical description of what is taking place within the computer. For convenience let $\nu^+ = z$ and $\nu^- = w$--equations (4) may then be written:

$$\text{i.}\quad \frac{\partial z}{\partial t} = D^+ \frac{\partial^2 z}{\partial x^2} - K^+(E_o + e_x)\frac{\partial z}{\partial x} - \left[\gamma_3 + 4\pi e K^+(z - w)\right] z + \gamma_4 w + 2.345 \text{ X } 10^{14}$$

$$\text{ii.}\quad \frac{\partial w}{\partial t} = D^- \frac{\partial^2 w}{\partial x^2} + K^-(E_o + e_x)\frac{\partial w}{\partial x} - \left[\gamma_1 - 4\pi e K^-(z - w)\right] w + \gamma_2 z + 2.345 \text{ X } 10^{14}$$

and from Poisson's Eqn,

$$\text{iii.}\quad \frac{\partial e_x}{\partial x} = 4\pi e(z - w)$$

Furthermore, we arbitrarily establish as an initial condition:

$$e_x = 0, \text{ at } x = 0$$

Let us now divide the part of our positive column wherein our equations will apply into n equal parts.

Let:

x_o denote the origin of our column (= 0)

x_1 be the first point at a distance Δx from x_o

then,

$x_i = i\Delta x$, where $i = 0, 1, \ldots\ldots, n$.

and,

$x = x_i - x_{i-1} = L/n$, L being the length of the column

Let:

$z_i = z(x_i,t)$

$w_i = w(x_i,t)$

$e_{x_i} = e_x(x_i) = 4\pi e(z_i - w_i)$ (from iii.)

Remembering that it was proposed to replace the partial derivatives with total derivatives (page 21), we now observe the technique of doing this by the finite approximation formulae:

$$\frac{\partial u}{\partial x}(x = x_i) \doteq \frac{u_{i+1} - u_i}{\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2}(x = x_i) \doteq \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

Utilizing this expediency on our first three equations in this chapter, we have:

i.a) $$\frac{dz_i}{dt} = \frac{D^+}{(\Delta x)^2}(z_{i+1} - 2z_i + z_{i-1}) - \frac{K^+}{\Delta x}(E_o + e_{x_i})(z_{i+1} - z_i) - \left[\gamma_3 + 4\pi e K^+(z_i - w_i)\right] z_i + \gamma_4 w_i + 2.345 \times 10^{14}$$

ii.a) $$\frac{dw_i}{dt} = \frac{D^-}{(\Delta x)^2}(w_{i+1} - 2w_i + w_{i-1}) + \frac{K^-}{\Delta x}(E_o + e_{x_i})(w_{i+1} - w_i) - \left[\gamma_1 - 4\pi e K^-(z_i - w_i)\right] w_i + \gamma_2 z_i + 2.345 \times 10^{14}$$

iii.a) $$e_{x_i} = e_{x_{i-1}} + 4\pi e(\Delta x)(z_i - w_i)$$

It is proposed to examine our solution over ten increments; therefore, we restrict our problem as follows:

$$i = 1, 2, 3, \ldots , 10$$

$$w_o = z_o = 0$$

$$w_{11} = z_{11} = 0$$

Rearranging equations i.a), ii.a), and iii.a) above, we have:

i.b) $$\frac{dz_i}{dt} = \left[\frac{D^+}{(\Delta x)^2} - \frac{K^+}{\Delta x}(E_o + e_{x_i})\right] z_{i+1}$$

$$+\left\{+\frac{K^{+}}{\Delta x}(E_o + e_{x_i}) - \frac{2D^{+}}{(\Delta x)^2} - \left[\gamma_3 + 4\pi eK^{+}(z_i - w_i)\right]\right\} z_i$$

$$+\left[\frac{D^{+}}{(\Delta x)^2}\right] z_{i-1} + \gamma_4 w_i + 2.345 \text{ X } 10^{14}$$

ii.b) $$\frac{dw_i}{dt} = \left[\frac{D^{-}}{(\Delta x)^2} + \frac{K^{-}}{\Delta x}(E_o + e_{x_i})\right] w_{i+1}$$

$$-\left\{\frac{K^{-}}{\Delta x}(E_o + e_{x_i}) + \frac{2D^{-}}{(\Delta x)^2} + \left[\gamma_1 - 4\pi eK^{-}(z_i - w_i)\right]\right\} w_i$$

$$+\left[\frac{D^{-}}{(\Delta x)^2}\right] w_{i-1} + \gamma_2 z_i + 2.345 \text{ X } 10^{14}$$

iii.b) $$e_{x_i} = e_{x_{i-1}} + 4\pi e(\Delta x)(z_i - w_i)$$

Let us now introduce some new constants by combining some of those which frequently occur; specifically let,

$$A_1 = \frac{D^{+}}{(\Delta x)^2} = .387 \quad \text{(arbitrarily setting } x = 3\text{cm)}$$

$$A_2 = \frac{K^{+}}{\Delta x} = 1.35 \text{ X } 10^4$$

$$A_3 = 4\pi eK^{+} = 2.44 \text{ X } 10^{-4}$$

$$B_1 = \frac{D^{-}}{(\Delta x)^2} = .720 \text{ X } 10^4$$

$$B_2 = \frac{K^{-}}{\Delta x} = .6 \text{ X } 10^7$$

$$B_3 = 4\pi eK^{-} = .109$$

Inserting these new constants into equations i.b), ii.b), and iii.b) yields:

i.c) $$\frac{dz_i}{dt} = \left[A_1 - A_2(E_o + e_{x_i})\right] z_{i+1}$$

$$+\left\{A_2(E_o + e_{x_i}) - 2A_1 - A_3\left[n_o^{-} + (z_i - w_i)\right]\right\} z_i$$

$$+A_1 z_{i-1} + A_3 n_o^{-} w_i + 2.345 \text{ X } 10^{14}$$

$$\text{ii.c)}\quad \frac{dw_i}{dt} = \left[B_1 - B_2(E_o + e_{x_i})\right]w_{i+1} - \left\{B_2(E_o + e_{x_i}) + 2B_1 + B_3\left[n_o^- - (z_i - w_i)\right]\right\}w_i + B_1 w_{i-1} + B_3 n_o^- z_i + 2.345 \times 10^{14}$$

$$\text{iii.c)}\quad e_{x_i} = e_{x_{i-1}} + 6.023 \times 10^{-9}(\Delta x)(z_i - w_i)$$

It should be noted that the value of n_o^- is arbitrarily taken to be 7×10^{10}. While this may or may not be exact, it is the correct order of magnitude. Further, in the ambipolar region $n_o^- = n_o^+$. In order to avoid overcomplicating the manipulation of decimals a new variable is introduced:

$$z_i = 10^{15} z_i'$$

$$w_i = 10^{15} w_i'$$

Changing variables and inserting the values of the constants where applicable:

$$\frac{dz_i'}{dt} = \left[.387 - .135 \times 10^5 (.02 + e_{x_i})\right] z_{i+1}' + \left\{.135 \times 10^5 (.02 + e_{x_i}) - .774 - .244 \times 10^{12}\left[.7 \times 10^{-4} + (z_i' - w_i')\right]\right\} z_i' + .387\, z_{i-1}' + .1708 \times 10^8\, w_i' + .2345$$

$$\frac{dw_i'}{dt} = \left[.720 \times 10^4 - .6 \times 10^7 (.02 + e_{x_i})\right] w_{i+1}' + \left\{.6 \times 10^7 (.02 + e_{x_i}) + 1.44 \times 10^4 + .109 \times 10^{15}\left[.7 \times 10^{-4} - (z_i' - w_i')\right]\right\} w_i' + .720 \times 10^4\, w_{i-1}' + .763 \times 10^{10}\, z_i' + .2345$$

$$e_{x_i} = e_{x_{i-1}} + .1807 \times 10^8 (z_i' - w_i')$$

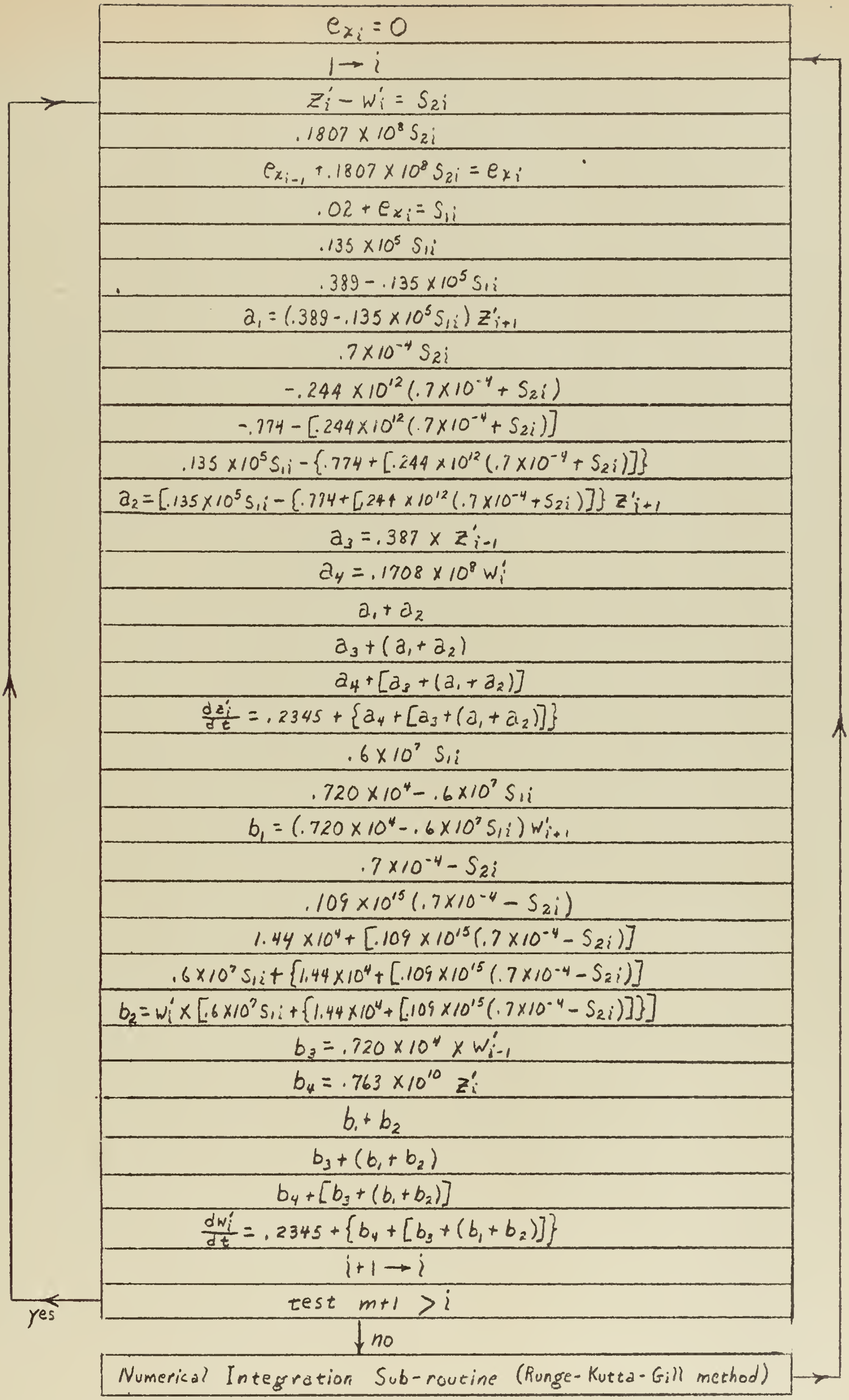
Let us define $s_{1i} = .02 + e_{x_i}$ and $s_{2_i} = z_i' - w_i'$, then the preceding equations can be written as:

$$\frac{dz_i'}{dt} = [.389 - .135 \times 10^5 s_{1i}] z_{i+1}' + \{.135 \times 10^5 s_{1i} - .774 - .244 \times 10^{12} [.7 \times 10^{-4} + s_{2i}]\} z_i'$$
$$+ .387 z_{i-1}' + .1708 \times 10^8 w_i' + .2345$$

$$\frac{dw_i'}{dt} = [.720 \times 10^4 - .6 \times 10^7 s_{1i}] w_{i+1}' + \{.6 \times 10^7 s_{1i} + 1.44 \times 10^4 + .109 \times 10^{15} [.7 \times 10^{-4} - s_{2i}]\} w_i'$$
$$+ .720 \times 10^4 w_{i-1}' + .763 \times 10^{10} z_i' + .2345$$

$$e_{x_i} = e_{x_{i-1}} + .1807 \times 10^8 s_{2i}$$

These are the final forms of the equations which the computer will be asked to solve. A general idea of the method of solution can be obtained by considering these equations in conjunction with the flow diagram which appears on the following page. The details of execution of the program are found following the flow diagram.


$e_{x_i} = 0$
$1 \rightarrow i$
$Z'_i - W'_i = S_{2i}$
$.1807 \times 10^8 S_{2i}$
$e_{x_{i-1}} + .1807 \times 10^8 S_{2i} = e_{x_i}$
$.02 + e_{x_i} = S_{1i}$
$.135 \times 10^5 S_{1i}$
$.389 - .135 \times 10^5 S_{1i}$
$a_1 = (.389 - .135 \times 10^5 S_{1i}) Z'_{i+1}$
$.7 \times 10^{-4} S_{2i}$
$-.244 \times 10^{12} (.7 \times 10^{-4} + S_{2i})$
$-.774 - [.244 \times 10^{12} (.7 \times 10^{-4} + S_{2i})]$
$.135 \times 10^5 S_{1i} - \{.774 + [.244 \times 10^{12} (.7 \times 10^{-4} + S_{2i})]\}$
$a_2 = [.135 \times 10^5 S_{1i} - \{.774 + [.244 \times 10^{12} (.7 \times 10^{-4} + S_{2i})]\} Z'_{i+1}$
$a_3 = .387 \times Z'_{i-1}$
$a_4 = .1708 \times 10^8 W'_i$
$a_1 + a_2$
$a_3 + (a_1 + a_2)$
$a_4 + [a_3 + (a_1 + a_2)]$
$\frac{dz'_i}{dt} = .2345 + \{a_4 + [a_3 + (a_1 + a_2)]\}$
$.6 \times 10^7 S_{1i}$
$.720 \times 10^4 - .6 \times 10^7 S_{1i}$
$b_1 = (.720 \times 10^4 - .6 \times 10^7 S_{1i}) W'_{i+1}$
$.7 \times 10^{-4} - S_{2i}$
$.109 \times 10^{15} (.7 \times 10^{-4} - S_{2i})$
$1.44 \times 10^4 + [.109 \times 10^{15} (.7 \times 10^{-4} - S_{2i})]$
$.6 \times 10^7 S_{1i} + \{1.44 \times 10^4 + [.109 \times 10^{15} (.7 \times 10^{-4} - S_{2i})]$
$b_2 = W'_i \times [.6 \times 10^7 S_{1i} + \{1.44 \times 10^4 + [.109 \times 10^{15} (.7 \times 10^{-4} - S_{2i})]\}]$
$b_3 = .720 \times 10^4 \times W'_{i-1}$
$b_4 = .763 \times 10^{10} Z'_i$
$b_1 + b_2$
$b_3 + (b_1 + b_2)$
$b_4 + [b_3 + (b_1 + b_2)]$
$\frac{dw'_i}{dt} = .2345 + \{b_4 + [b_3 + (b_1 + b_2)]\}$
$i+1 \rightarrow i$
test $m+1 > i$
yes
no
Numerical Integration Sub-routine (Runge-Kutta-Gill method)

The following is the detailed program of the computer solution to our problem.

Loc	Opn	Opn Code	M_1	M_2	M_3	Remarks
0010	md	26	2100	2100	0205	$0 \rightarrow e_{x_i}$
0011	ad	35	2100	0201	0200	$1 X 2^{-12} + 1 X 2^{-24} + i X 2^{-12} + i X 2^{-24}$
0012	ad	35	0200	0200	0207	
0013	ad	35	0210	0207	0014	
0014	su	36	[z_i'	w_i'	0202]	$z_i' - w_i' = s_{2i}$ in 0202
0015	tm	34	0202	2100	0020	no → 0016, yes → 0020
0016	md	26	2100	2100	2000	$0 \rightarrow \left\{\begin{matrix}2000\\2001\end{matrix}\right.$
0017	tm	34	3000	2100	0022	
0020	sf	31	0240	0202	2000	$.1807 X 10^8 X s_{2i}$ in $\left\{\begin{matrix}2000\\2001\end{matrix}\right.$
0021	mr	25	0241	2001	2001	
0022	ad	35	2100	0205	2002	$e_{x_{i-1}}$ in $\left\{\begin{matrix}2002\\2003\end{matrix}\right.$
0023	ad	35	2100	0206	2003	
0024	tm	34	3000	2100	0400	gives e_{x_i} in $\left\{\begin{matrix}2000\\2001\end{matrix}\right.$
0025	ad	35	2100	0242	2002	.02 in $\left\{\begin{matrix}2002\\2003\end{matrix}\right.$
0026	ad	35	2100	0243	2003	
0027	tm	34	3000	2100	0400	$.02 + e_{x_i} = s_{1i}$ in $\left\{\begin{matrix}2000\\2001\end{matrix}\right.$
0030	ad	35	2100	2000	0203	s_{1i} in 0203 and 0204
0031	ad	35	2100	2001	0204	
0032	tm	34	2001	2100	0035	no → 0033, yes → 0035
0033	md	26	2100	2100	2000	0 → 2000, 2001
0034	tm	34	3000	2100	0037	
0035	ad	35	2000	0244	2000	$-.135 X 10^5 s_{1i}$
0036	mr	25	2001	0245	2001	

```
0037  ad  35  2100  0246  2002
0040  ad  35  2100  0247  2003    .389 in 2002 and 2003
0041  tm  34  3000  2100  0400    .389 - .135 X 10^5 s_1i in {2000
                                                             {2001
0042  tm  34  2001  2100  0045    no→0043, yes→0045
0043  md  26  2100  2100  2000
0044  tm  34  3000  2100  0054
0045  ex  32  0014  0211  0212
0046  ad  35  0212  0213  2004
0047  ex  32  2004  0211  0050
0050  tm  34 [z'_i÷1] 2100 0053    no→0051, yes→0053
0051  md  26  2100  2100  2000
0052  tm  34  3000  2100  0054
0053  tm  34  3000  2100  0500    a_1 in {2000
                                         {2001
0054  ad  35  2100  2000  0214    a_1 in {0214
                                         {0215
0055  ad  35  2100  2001  0215
0056  ad  35  2100  2100  2000}
                                }  s_2i in 2000 and 2001
0057  ad  35  2100  0202  2001}
0060  ad  35  2100  0250  2002}
                                }  .7 X 10^-4 in 2002 and 2003
0061  ad  35  2100  0251  2003}
0062  tm  34  3000  2100  0400    .7 X 10^-4 ÷ s_2i in 2000 and 2001
0063  tm  34  2000  2100  0066    no→0064, yes→0066
0064  md  26  2100  2100  2000
0065  tm  34  3000  2100  0072
0066  ad  35  0252  2000  2000}
                                }  .244 X 10^12 (.7 X 10^-4 ÷ s_2i)
0067  mr  25  0253  2001  2001}
0070  ad  35  2100  0254  2002}
                                }  .774 in 2002 and 2003
0071  ad  35  2100  0255  2003}
```

```
0072  tm  34  3000    2100  0400    -.774 -.244 X 10^12 (.7 X 10^-4 + s_zi) in
                                    2000 and 2001
0073  ex  32  0212    2211  0074
0074  tm  34  [ z'_i ]  2100  0077  no -> 0075, yes -> 0077
0075  md  26  2100    2100  2000
0076  tm  34  3000    2100  0101
0077  tm  34  3000    2100  0510
0100
0101  ad  35  2100    2000  0216 \           (0216
                                  }  a_2 in {
0102  ad  35  2100    2001  0217 /           (0217
0103  ad  35  2100    0257  2001    .387 in 2001
0104  ad  35  2100    0212  2004
0105  su  36  2004    0213  2004
0106  ex  32  2004    0211  0107
0107  tm  34  [z'_i-1]  2100  0112  no -> 0110, yes -> 0112
0110  md  26  2100    2100  2000
0111  tm  34  3000    2100  0114
0112  tm  34  3000    2100  0516
0113
0114  ad  35  2100    2100  0220    a_3 in 0220 and 0221
0115  ad  35  2100    2001  0221    a_3 in 0220 and 0221
0116  ad  35  0212    0226  2004
0117  ex  32  2004    0211  0120
0120  tm  34  [ w'_i ]  2100  0123  no -> 0121, yes -> 0123
0121  md  26  2100    2100  2000
0122  tm  34  3000    2100  0125
0123  tm  34  3000    2100  0524
```

0124						
0125	ad	35	2100	2000	0222	a_4 in 0222, 0223
0126	ad	35	2100	2001	0223	
0127	ad	35	2100	0220	2002	a_3 in 2002, 2003
0130	ad	35	2100	0221	2003	
0131	tm	34	3000	2100	0400	$a_3 \not+ a_4$ in 2000, 2001
0132	ad	35	2100	0216	2002	a_2 in 2002, 2003
0133	ad	35	2100	0217	2003	
0134	tm	34	3000	2100	0400	$a_2 \not+ a_3 \not+ a_4$ in 2000, 2001
0135	ad	35	2100	0214	2002	a_1 in 2002, 2003
0136	ad	35	2100	0215	2003	
0137	tm	34	3000	2100	0400	$a_1 \not+ a_2 \not+ a_3 \not+ a_4$ in 2000, 2001
0140	ad	35	2100	0262	2002	.2345 in 2002, 2003
0141	ad	35	2100	0263	2003	
0142	tm	34	3000	2100	0400	$\frac{dz_i'}{dt}$ in 2000, 2001
0143	ad	35	2000	0227	2000	$\frac{dz_i'}{dT}$ ($\frac{dz_i}{dt}$ scaled) in 2000, 2001
0144	tm	34	3000	2100	0145	
0145	ta	33	0157	2000	0150	
0146	pr	21	2000	0147	0002	
0147	ht	22	f	f	f	
0150	sm	30	2001	2000	2002	$\frac{dz_i'}{dT}$ in 2002
0151	sm	30	0014	0227	2004	$[z_i]$ in m_3 of 2004
0152	ex	32	2004	0156	0154	
0153	ad	35	0154	0277	0154	$[\dot{z}_i']$ in m_3 of 0122
0154	ad	35	2002	2100	f	$\frac{dz_i'}{dT}$ transferred to integration sub-routine
0155	tm	34	3000	2100	0160	

```
0156                 f     f    7777
0157                 f     f    0001
0160  tm    34    0204  2100    0163      no → 0161, yes → 0163
0161  md    26    2100  2100    2000
0162  tm    34    3000  2100    0170
0163  ad    35    0203  0264    2000 ┐
0164  mr    25    0204  0265    2001 ├    .6 X 10^7 s_{1i} in 2000, 2001
0165  su    36    2100  2001    2001 ┘
0166  ad    35    2100  0266    2002 ┐
                                     ├    .720 X 10^4 in 2002, 2003
0167  ad    35    2100  0267    2003 ┘
0170  tm    34    3000  2100    0400      .720 X 10^4 -.6 X 10^7 s_{1i} in 2000, 2001
0171  tm    34    2001  2100    0174      no → 0172, yes → 0174
0172  md    26    2100  2100    2000
0173  tm    34    3000  2100    0311
0174  ad    35    0120  0213    2004
0175  ex    32    2004  0211    0304
0176  tm    34    3000  2100    0304
0177
0200                                      i X 2^{-12} + i X 2^{-24}
0201              0001  0001     f
0202                                      s_{2i} temp storage
0203                                    ┐
                                        ├ s_{1i}
0204                                    ┘
0205                                    ┐
                                        ├ e_{x_i}
0206                                    ┘
0207                                      2 X (i X 2^{-12} + i X 2^{-24})
```

```
0210  su   36   1305  1306  0202     dummy for 0014

0211            7777     f     f     extracted for 0034

0212  mr   25      f  2001  2001     dummy for 0036

0213            0002     f     f

0214                                ⎫
                                    ⎬ a1 temporary storage
0215                                ⎭

0216                                ⎫
                                    ⎬ a2 temporary storage
0217                                ⎭

0220                                ⎫
                                    ⎬ a3 temporary storage
0221                                ⎭

0222                                ⎫
                                    ⎬ a4 temporary storage
0223                                ⎭

0224                                ⎫
                                    ⎬ dz'i/dt temporary storage
0225                                ⎭

0226            0001     f     f

0227       02      f     f  0030

0230                                ⎫
                                    ⎬ b1 storage
0231                                ⎭

0232                                ⎫
                                    ⎬ b2 storage
0233                                ⎭

0234                                ⎫
                                    ⎬ b3 storage
0235                                ⎭

0236                                ⎫
                                    ⎬ b4 storage
0237                                ⎭

0240               f     f  0033    ⎫
                                    ⎬ .1807 X 10^8
0241            1047  3476  0000    ⎭
```

```
0242           f     f     f   }
                                 .02
0243        0121  7300  0000   }
0244           f     f  0017   }
                                 -.135 X 10^5
0245   02   3227  3777  7720   }
0246           f     f     f   }
                                 .389
0247        3071  2601  0142   }
0250           f     f     f   }
                                 .7 X 10^-4
0251        0000  2226  3173   }
0252           f     f  0047   }
                                 -2.44 X 10^12
0253   02   3441  3567  6333   }
0254           f     f     f   }
                                 -.774
0255   02   6142  2335  1362   }
0256           f     f     f   }
                                  .387
0257        3061  1156  4571   }
0260           f     f  0033   }
                                  .1708 X 10^8
0261        1011  1730  0000   }
0262           f     f     f   }
                                  .2345
0263        1700  4061  1156   }
0264           f     f  0030   }
                                  .6 X 10^7
0265        2670  6600  0000   }
0266        0000  0000  0017   }
                                  .720 X 10^4
0267        1604  0000  0000   }
0270           f     f  0060   }
                                  .109 X 10^15
0271        3062  1212  2176   }
0272           f     f  0017   }
                                 1.44 X 10^4
0273        3407  7777  4512   }
```

```
0274                  f      f    0041 ⎫
                                       ⎬  .763 X 10^10
0275               7066   2107    7440 ⎭

0276               0013   0013      f     m +1  m +1 f, m = 12 (octal)

0277                  f      f    0027    (2m+1) X 2^-36

0300

0301

0302

0303

0304   tm   34   [w'i+1]  2100    0307    no → 0305, yes → 0307

0305   md   26     2100   2100    2000

0306   tm   34     3000   2100    0311

0307   tm   34     3000   2100    0540

0310

0311   ad   35     2100   2000    0230 ⎫
                                       ⎬  b_1 in 0230, 0231
0312   ad   35     2100   2001    0231 ⎭

0313   su   36     2100   0202    2001    s_2i in 2001

0314   ad   35     2100   2100    2000

0315   ad   35     2100   0250    2002 ⎫
                                       ⎬  .7 X 10^-4 in 2002, 2003
0316   ad   35     2100   0251    2003 ⎭

0317   tm   34     3000   2100    0400    .7 X 10^-4 - s_2i in 2000, 2001

0320   tm   34     2001   2100    0323    no →321, yes →323

0321   md   26     2100   2100    2000

0322   tm   34     3000   2100    0330

0323   ad   35     2000   0270    2000 ⎫
                                       ⎬  .109 X 10^15 X (.7 X 10^-4 - s_2i)
0324   mr   25     2001   0271    2001 ⎭

0325   ad   35     2100   0272    2002 ⎫
                                       ⎬  1.44 X 10^4 in 2002, 2003
0326   ad   35     2100   0273    2003 ⎭
```

```
0327  tm  34  3000    2100  0400    1.44 X 10^4 + .109 X 10^15 X (.7 X 10^-4-s_2i)
                                    in 2000, 20001
0330  tm  34  0204    2100  0333    no → 0331, yes → 0333

0331  md  26  2100    2100  2002

0332  tm  34  3000    2100  0335

0333  ad  35  0203    0264  2002

0334  mr  25  0204    0265  2003

0335  tm  34  3000    2100  0400

0336  ex  32  0120    0211  0337

0337  tm  34  [w'_i]  2100  0342    no → 0340, yes → 0342

0340  md  26  2100    2100  2000  }
                                  }
0341  tm  34  3000    2100  0344  }
                                  }
0342  tm  34  3000    2100  0550  }
                                  }  b_2 in 0232, 0233
0343                              }
                                  }
0344  ad  35  2100    2000  0232  }
                                  }
0345  ad  35  2100    2001  0233  }

0346  ad  35  2100    0266  2000  }
                                  }  .720 X 10^4 in 2000, 2001
0347  ad  35  2100    0267  2001  }

0350  su  36  0120    0213  2004

0351  ex  32  2004    0211  0352

0352  tm  34  [w'_i-1] 2100 0355    no → 0353, yes → 0355

0353  md  26  2100    2100  2000

0354  tm  34  3000    2100  0357

0355  tm  34  3000    2100  0560

0356

0357  ad  35  2100    2000  0234  }
                                  }  b_3 in 0234, 0235
0360  ad  35  2100    2001  0235  }
```

```
0361  ad  35  2100    0274  2000 ⎫
                                 ⎬  .763 X 10^10 in 2000, 2001
0362  ad  35  2100    0275  2001 ⎭

0363  ex  32  0074    0211  0364

0364  tm  34  [ z'i ] 2100  0367    no →0365, yes →0367

0365  md  26  2100    2100  2000

0366  tm  34  3000    2100  0371

0367  tm  34  3000    2100  0570

0370

0371  ad  35  2100    2000  0236 ⎫
                                 ⎬  b4 in 0236, 0237
0372  ad  35  2100    2001  0237 ⎭

0373  ad  35  2100    0234  2002 ⎫
                                 ⎬  b3 in 2002, 2003
0374  ad  35  2100    0235  2003 ⎭

0375  tm  34  3000    2100  0400    b3 ≠ b4 in 2000, 2001

0376  tm  34  3000    2100  0440

0377

0400  sl  27  3000    0422  2004

0401  ad  35  0423    2004  0410

0402  tm  34  3000    2100  0426

0403  su  36  2002    2000  2004

0404  sm  30  2003    2004  2003

0405  ad  35  2003    2001  2001

0406  to  37  2001    3000  0415

0407  sf  31  2000    2001  2000

0410                                exit

0411  su  36  2000    2002  2004

0412  sm  30  2001    2004  2001
```

```
0413  ad  35  2002  2100  2000
0414  tm  34  3000  2100  0405
0415  sl  27  2001  0424  2001
0416  sm  30  2001  0424  2001
0417  sl  27  2001  0425  2001
0420  ad  35  0425  2000  2000
0421  tm  34  3000  2100  0407
0422      02    f     f   0030
0423  tm  34  3000  2100    f
0424      02    f     f   0001
0425            f     f   0001
0426  tm  34  2001  2100  0431
0427  tm  34  2003  2100  0433
0430  tm  34  3000  2100  0410
0431  sf  31  2000  2001  2000
0432  tm  34  3000  2100  0427
0433  sf  31  2002  2003  2002
0434  ta  33  2002  2000  0411
0435  tm  34  3000  2100  0403
0436
0437
0440  ad  35  2100  0232  2002 ⎫
                               ⎬  b2 in 2002, 2003
0441  ad  35  2100  0233  2003 ⎭
0442  tm  34  3000  2100  0400    b2 ≠ b3 ≠ b4 in 2000, 2001
0443  ad  35  2100  0230  2002 ⎫
                               ⎬  b1 in 2002, 2003
0444  ad  35  2100  0231  2003 ⎭
```

```
0445  tm  34  3000  2100  0400    b1 + b2 + b3 + b4 in 2000, 2001

0446  ad  35  2100  0262  2002 )
                                }  .2345 in 2002, 2003
0447  ad  35  2100  0263  2003 )

0450  tm  34  3000  2100  0400    dw'i/dt  in 2000, 2001

0451  ad  35  2000  0227  2000    dw'i/dT

0452  ta  33  0157  2000  0455

0453  pr  21  2000  0454  0002

0454  ht  22    f     f     f

0455  sm  30  2001  2000  2002

0456  sm  30  0337  0227  2004

0457  ex  32  2004  0156  0461

0460  ad  35  0461  0277  0461

0461  ad  35  2002  2100    f     dw'i/dT  transferred to integration sub-routine

0462  ad  35  0201  0200  0200    (i+1) X 2^-12+(i+1) X 2^-24→iX2^-12+
                                    iX2^-24
0463  tm  34  0276  0200  0012    test m + 1 > i

0464  tm  34  3000  2100  1100    U.T. to integration sub-routine

0465  tm  34  3000  2100  0010

0500  sm  30  0050  0506  2005    [z'i+1]  in m2 of 2005

0501  ex  32  2005  0507  0502

0502  sf  31  2100 [z'i+1] 2002

0503  ad  35  2000  2002  2000 )
                                }  a1 in 2000, 2001
0504  mr  25  2001  2003  2001 )

0505  tm  34  3000  2100  0054

0506      02    f     f   0014

0507          f   7777    f

0510  sm  30  0212  0506  2005
```

```
0511  ex   32   2005   0507    0512

0512  sf   31   2100  [z'i]    2002

0513  ad   35   2000   2002    2000 \
                                     }  a2 in 2000, 2001
0514  mr   25   2001   2003    2001 /

0515  tm   34   3000   2100    0101

0516  sm   30   0107   0506    2005

0517  ex   32   2005   0507    0520

0520  sf   31   2100  [z'i-1]  2002

0521  ad   35   2000   2002    2000 \
                                     }  a3 in 2000, 2001
0522  mr   25   2001   2003    2001 /

0523  tm   34   3000   2100    0114

0524  sm   30   0120   0506    2005

0525  ex   32   2005   0507    0526

0526  sf   31   2100  [w'i]    2002

0527  ad   35   2000   2002    2000 \
                                     }  a4 in 2000, 2001
0530  mr   25   2001   2003    2001 /

0531  tm   34   3000   2100    0125

(cells 0532 through 0537 are filled with zeros)

0540  sm   30   0304   0506    2005

0541  ex   32   2005   0507    0542

0542  sf   31   2100  [w'i+1]  2002

0543  ad   35   2000   2002    2000 \
                                     }  b1 in 2000, 2001
0544  mr   25   2001   2003    2001 /

0545  tm   34   3000   2100    0311

0546

0547
```

```
0550  sm   30   0337  0506     2005
0551  ex   32   2005  0507     0552
0552  sf   31   2100  [w'_i]   2002
0553  ad   35   2000  0002     2000 }
                                    }  b_2 in 2000, 2001
0554  mr   25   2001  2003     2001 }
0555  tm   34   3000  2100     0344
0556
0557
0560  sm   30   0352  0506     2005
0561  ex   32   2005  0507     0562
0562  sf   31   2100  [w'_i-1] 2002
0563  ad   35   2000  2002     2000 }
                                    }  b_3 in 2000, 2001
0564  mr   25   2001  2003     2001 }
0565  tm   34   3000  2100     0357
0566
0567
0570  sm   30   0364  0506     2005
0571  ex   32   2005  0507     0572
0572  sf   31   2100  [z'_i]   2002
0573  ad   35   2000  2002     2000
0574  mr   25   2001  2003     2001
0575  tm   34   3000  2100     0371
0576
0577
```

```
1000  sl   27   3000   1016   2001
1001  ad   35   1014   2001   1013
1002  ad   35   2000   2100   1022
1003  bl   05   3000   3000   1014
1004  ex   32   2002   1000   2000
1005  mr   25   2002   2007   2002
1006  ex   32   2002   2001   2000
1007  ex   32   2100   2001   2002
1010  sl   27   2001   2005   2001
1011  tm   34   2001   2100   1005
1212  pr   21   2000   1020   0001
1013
1014  tm   34   3000   2100     f
1015       02     f      f    0004
1016       02     f      f    0030
1017            5000     f      f
1020       01     f      f      f
1021            7400     f      f
```

(cells 1022 through 1047 are filled with zeros)

```
1050  ad   35   2100   2100   1057    clear i tally
1051  ad   35   1060   1057   1052
1052 [ad   35   1305+i 2100   2000]
1053  tm   34   3000   2100   1000    U.T. to convert and print sub-routine
1054  ad   35   1062   1057   1057    i + 1 → i
1055  tm   34   1061   1057   1051
1056  tm   34   3000   2100   1160
```

```
1057                                $i \times 2^{-12}$
1060  ad   35   1305   2100   2000
1061            0030    f      f    $m + 1$ ff
1062            0001    f      f
```

(The following is the sub-routine for the integration of <u>n</u> 1st order equations by the Runge-Kutta-Gill method)

```
1100  sl   27   3000   1242   2000 \
                                    } Prepare exit  to main program
1101  ad   35   1246   2000   1160 /
1102  tm   34   1163   1303   1200  Test if first time through
1103  tm   34   3000   2100   1107  U. T.
1104  ad   35   2100   2100   1247  0 → k (clear print tally)
1105  ad   35   2100   2100   1250  0 → j (clear iteration tally)
1106  ad   35   2100   1177   1251  $1 \times 2^{-12}$ (reset var. no. tally)
1107  ad   35   1250   1251   1252  $i \times 2^{-12} + j \times 2^{-24}$ tally
1110  ad   35   1234   1252   1111  $[q_i^j]$ in $M_1$ and $[B^j]$ in $M_2$ of 1111
1111 [sm   30   $q_i^j$  $B^j$  2001]  $B^j \times q_i^j$
1112  ad   35   1232   1251   1113  $[\dot{x}_i^j]$ in $M_1$ of 1113
1113 [su   36   $\dot{x}_i^j$  2001   2004]  $\dot{x}_i^j - B^j q_i^j$
1114  ad   35   1267   1250   1115  $[A^j]$ in $M_2$ of 1115
1115 [mr   25   2004   $A^j$  2005]  $A^j(\dot{x}_i^j - B^j q_i^j)$
1116  su   36   2004   2005   2006  $(1-A^j)(\dot{x}_i^j - B^j q_i^j) = r_i^j = \Delta x_i^j/\tau$
1117  to   37   2006   2100   1226  test $|r_i^j| \geq 1$ (truncation test)
1120  mr   25   2006   1301   2007  $(1-A^j)(\dot{x}_i^j - B^j q_i^j)\tau = \Delta x_i^j$
1121  sm   30   1251   1242   1270  $i \times 2^{-36}$ from $i \times 2^{-12}$
1122  ad   35   1251   1270   1271  $i \times 2^{-12} + i \times 2^{-36}$
1123  ad   35   1272   1271   1124  $[x_i^j]$ in $M_1$ and $M_3$ of 1124
```

1124	[ad	35	X_1^j	2007	x_i^j]	X_i^j X ΔX_i^j = $X_i^j + 1 \longrightarrow X_i^j$
1125	sm	30	1251	1235	2003	i X 2^{-24} from i X 2^{-12}
1126	ad	35	1251	2003	1273	i X 2^{-12} + i X 2^{-24}
1127	ad	35	1237	1273	1130	$[q_i^j]$ in M_1 and $[\dot{x}_i^j]$ in M_2 of 1130
1130	[su	36	q_1^j	$\dot{x}_1^j$	2007]	$q_i^j - \dot{x}_i^j$
1131	ad	35	1241	1252	1132	$[\dot{x}_i^j]$ in M_1 and $[C^j]$ in M_2 of 1132
1132	[mr	25	$\dot{X}_1^j$	c^j	2000]	$\dot{x}_i^j \times C^j$
1133	sm	30	2007	1274	2007	$(q_i^j - \dot{x}_i^j)$ X 8^{-1}
1134	sm	30	2000	1274	2000	$\dot{x}_i^j \times C^j$ X 8^{-1}
1135	ad	35	2000	2007	2001	$[q_i^j + (c^j - 1)\ \dot{x}_i^j]$ X8^{-1}
1136	mr	25	2006	1275	2006	$3\ r_i^j$ X 8^{-1}
1137	ad	35	2001	2006	2002	$[q_i^j + (c^j - 1)\dot{x}_i^j] \times 8^{-1} + 3 r_i^j \times 8^{-1} = q_i^{j+1} \times 8^{-1}$
1140	sm	30	1111	1242	2003	$[q_i^{j+1}]$ in M_3 of 2003
1141	ad	35	1276	2003	1142	$[q_i^{j+1}]$ in M_3 of 1142
1142	[sm	30	2002	1277	q_i]	$(q_i^{j+1} \times 8^{-1})\,8 \rightarrow q_i^j$
1143	ad	35	1177	1251	1251	(i + 1) X 2^{-12} → i X 2^{-12}
1144	tm	34	1300	1251	1107	test (n + 1) > i
1145	ad	35	2100	1177	1251	1 X 12^{-12} → i X 2^{-12}
1146	ts	17	2010	3000	1164	print j X 2^{-24} if ts 2010 up
1147	ad	35	1250	1175	1250	(i + 1) X 2^{-24} → j X 2^{-24}
1150	tm	34	1176	1250	1160	test 4 > j
1151	ad	35	2100	2100	1250	0 → j X 2^{-24}
1152	ts	17	2020	3000	1166	accumulate τ if 2020 up
1153	ad	35	1163	1247	1247	k + 1 → k
1154	tm	34	1302	1247	1160	test K > k
1155	ad	35	2100	2100	1247	0 → k
1156	ts	17	2040	3000	1170	print X_1 octal if 2040 up

1157	tm	34	3000	2100	1050	U. T. to new print routine
1160						U. T. to $\dot{X}_1$ SR
1161						
1162						
1163			f	f	0001	$1 X 2^{-36}$
1164	pr	21	1250	2100	0001	print $j X 2^{-24}$
1165	tm	34	3000	2100	1147	U. T.
1166	ad	35	1301	1304	1304	$X_0 + (m + 1)\tau \rightarrow X_0 + m\tau = t$
1167	tm	34	3000	2100	1153	U. T.
1170	ts	17	2020	3000	1173	
1171	[pr	21	1305	2100	n]	print $X_1, X_2, \ldots, X_n$
1172	tm	34	3000	2100	1160	
1173	[pr	21	1304	2100	n + 1]	print $X_0 = t, X_1, X_2, \ldots X_n$
1174	tm	34	3000	2100	1160	
1175			f	0001	f	$1 X 2^{-24}$
1176			f	0004	f	$4 X 2^{-24}$
1177			0001	f	f	$1 X 2^{-12}$
1200	ad	35	1230	1300	2000	$[\dot{x}_0]$ in M_1 of 2000
1201	ad	35	1231	2000	1232	$[\dot{x}_0]$ in M_1 of mod. dummy 1
1202	ad	35	1300	2000	2002	$[q_0]$ in M_1 of 2002
1203	ad	35	1233	2002	1234	$[q_0]$ in M_1 of mod. dummy 2
1204	sm	30	2000	1235	2003	$[\dot{x}_0]$ in M_2 of 2003
1205	ad	35	2002	2003	2004	$[q_0]$ in M_1 and $[\dot{x}_0]$ in M_2 of 2004
1206	ad	35	1236	2004	1237	$[q_0]$ in M_1 and $[\dot{x}_0]$ in M_2 of mod.dummy 3
1207	ad	35	1240	2000	1241	$[\dot{x}_0]$ in M_1 of mod. dummy 4
1210	sm	30	2002	1242	2002	$[q_0]$ in M_3 of 2002

1211	sm	30	1300	1242	2000	$n + 1 X 2^{-36}$ in 2000
1212	ad	35	1243	2000	1173	sets up print of X_o, X_1, ... X_n
1213	su	36	2000	1163	2001	$n X 2^{-36}$ in 2001
1214	ad	35	1244	2001	1171	sets up print of X_1, X_2 , ... X_n
1215	ad	35	2100	2100	1270	$0 \rightarrow i x 2^{-36}$
1216	ad	35	2002	1270	2005	$[q_i]$ in M_3 of 2005
1217	ad	35	1245	2005	1220	$[q_i]$ in M_3 of 1220
1220	[ad	35	2100	2100	q_i]	$0 \rightarrow q_i$
1221	ad	35	1163	1270	1270	$(i+1) X 2^{-36} \rightarrow i X 2^{-36}$
1222	tm	34	1270	2000	1224	test $i > n+1$
1223	tm	34	3000	2100	1216	
1224	ad	35	2100	1163	1303	signal for first time through
1225	tm	34	3000	2100	1104	U. T.
1226	pr	21	2006	1227	0001	print r_i^j
1227	ht	22	f	f	f	halt
1230			1304	f	f	$[X_o] X 2^{-12}$
1231	su	36	f	2001	2004	dummy 1
1232	[su	36	$\dot{X}_o$	2001	2004]	modified dummy 1 for 1112
1233	sm	30	f	1253	2001	dummy 2
1234	[sm	30	q_o	1253	2001]	modified dummy 2 for 1111
1235		02	f	f	0014	
1236	su	36	f	f	2007	dummy 3
1237	[su	36	q_o	$\dot{X}_o$	2007]	modified dummy 3 for 1130
1240	mr	25	f	1263	2000	dummy 4
1241	[mr	25	$\dot{X}_o$	1263	2000]	modified dummy 4
1242		02	f	f	0030	

1243	pr	21	1304	2100	f	dummy print for 1173
1244	pr	21	1305	2100	f	dummy print for 1171
1245	ad	35	2100	2100	f	dummy for 1220
1246	tm	34	3000	2100	f	dummy exit command
1247						$k \times 2^{-36}$ (print tally)
1250						$j \times 2^{-24}$ (iteration tally)
1251						$i \times 2^{-12}$ (var. no. tally)
1252						$i \times 2^{-12} + j \times 2^{-24}$
1253			f	f	0001	B^0
1254			f	f	f	B^1
1255			f	f	f	B^2
1256			f	f	0001	B^3
1257			4000	f	f	$A^0 = 1/2$
1260			5520	2357	4151	$A^1 = 1/\sqrt{2}$
1261		02	5520	2357	4151	$A^2 = 1/\sqrt{2}$
1262			6525	2525	2523	$A^3 = 5/6$
1263			4000	f	f	$C^0 = 1/2$
1264			5520	2357	4151	$C^1 = 1/\sqrt{2}$
1265		02	5520	2357	4151	$C^2 = -1/\sqrt{2}$
1266			4000	f	f	$C^3 = 1/2$
1267	mr	25	2004	1257	2005	dummy for 1115
1270						$i \times 2^{-36}$
1271						$i \times 2^{-12} + i \times 2^{-36}$
1272	ad	35	1304	2007	1304	dummy for 1124
1273						$i \times 2^{-12} + i \times 2^{-24}$
1274		02	f	f	0003	
1275			3000	f	f	3×8^{-1}

```
1276  sm   30   2002   1277    f
1277                f       f    0003
```

(Cells 1300 through 1332 were filled by tape. These cells contain the values of t, z_i, w_i, etc. These values are not included here because they depend upon the values of computed parameters--see Chapter V.)

Some notes on the program:

All addresses appearing in brackets are not put in by punch cards but rather by other parts of the program. They are included for purposes of clarity in deciphering the program.

The following notes apply to the integration sub-routine:

ts 2010 -- up for iteration number tally

ts 2020 -- up for generation of x_o

ts 2040 -- up for print of x_i octal

n = no. of dependent variables (equations)

τ = incremental change of independent variable

K = readout at every K point

x_o = independent variable

New print or readout routine to start at 1050.

Automatic print of cell 2006 indicates interval of integration too large.

Error of integration is proportional to the 5th power of τ.

CHAPTER V

CONFLUDING REMARKS AND RECOMMENDATIONS FOR FURTHER WORK

As noted previously, it would be fatuous at this point to draw any definite conclusions anent the solution to our problem proposed in the first chapter. On the basis of the computer solutions we cannot say with absolute certainty that the results are compatible with experimental observations. We can say, however, on the basis of our sketchy results that the theory and method are promising. Without equivocation we can say that the computer program is eminently successful--it does work and it does solve the equations proposed as a description of the traveling density waves. A great deal of work is forecast in order to establish the identity of these equations with the phenomenon in the positive column. Great numbers of initial conditions should be inserted in the program and the coefficients should be varied in order to eventually establish the validity of the equations. It is believed that if this is done, the resulting solutions by the computer will correspond to the observations made under actual conditions in the laboratory. However, no precise conclusions can be drawn from a very few computations. These computations cannot be made in a few hours. It is predicted that the collection of enough data will require on the order of several hundred hours of computer time. The important point to be remembered is that a workable program is available and patience and perseverance will reward the eventual researcher with the answer. It appears from initial computer runs that the term zn_0^- is dominated in the equations (4) by other terms when $\nu^{\pm} \doteq n_0$. The initial computer run's suggest that "floating point arithmetic" might

be necessary throughout the program due to the drastic changes noted in derivatives and the time required for computation using the present program. No other trends were noted from the initial computer runs.

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